



Bounding multiplicative energy by the sumset[☆]

József Solymosi

University of British Columbia, Mathematics, 1984 Mathematics Road, Vancouver, British Columbia, V6T 1Z2, Canada

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Abstract

We prove that the sumset or the productset of any finite set of real numbers, A , is at least $|A|^{4/3-\varepsilon}$, improving earlier bounds. Our main tool is a new upper bound on the multiplicative energy, $E(A, A)$.
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1. Introduction

The *sumset* of a finite set of an additive group, A , is defined by

$$A + A = \{a + b : a, b \in A\}.$$

The *productset* and *ratio set* are defined in a similar way,

$$AA = \{ab : a, b \in A\},$$

and

$$A/A = \{a/b : a, b \in A\}.$$

A famous conjecture of Erdős and Szemerédi [5] asserts that for any finite set of integers, M ,

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E-mail address: solymosi@math.ubc.ca.

$$\max\{|M + M|, |MM|\} \geq |M|^{2-\varepsilon},$$

where $\varepsilon \rightarrow 0$ when $|M| \rightarrow \infty$. They proved that

$$\max\{|M + M|, |MM|\} \geq |M|^{1+\delta},$$

for some $\delta > 0$. In a series of papers, lower bounds on δ were found. $\delta \geq 1/31$ [10], $\delta \geq 1/15$ [6], $\delta \geq 1/4$ [3], and $\delta \geq 3/11$ [12]. The last two bounds were proved for finite sets of real numbers.

2. Results

Our main result is the following.

Theorem 2.1. *Let A be a finite set of positive real numbers. Then*

$$|AA||A + A|^2 \geq \frac{|A|^4}{4 \lceil \log |A| \rceil}$$

holds.

The inequality is sharp—up to the power of the log term in the denominator—when A is the set of the first n natural numbers. Theorem 2.1 implies an improved bound on the sum-product problem.

Corollary 2.2. *Let A be a finite set of positive real numbers. Then*

$$\max\{|A + A|, |AA|\} \geq \frac{|A|^{4/3}}{2 \lceil \log |A| \rceil^{1/3}}$$

holds

2.1. Proof of Theorem 2.1

To illustrate how the proof goes, we are making two unjustified and usually false assumptions, which are simplifying the proof. Readers, not interested in this “handwaving”, will find the rigorous argument about 20 lines below.

Suppose that AA and A/A have the same size, $|AA| \approx |A/A|$, and any element of A/A has about the same number of representations as any other. This means that for any reals $s, t \in A/A$ the two numbers s and t have the same multiplicity, $|\{(a, b) \mid a, b \in A, a/b = s\}| \approx |\{(b, c) \mid b, c \in A, b/c = t\}|$. A geometric interpretation of the cardinality of A/A is that the Cartesian product $A \times A$ is covered by $|A/A|$ concurrent lines going through the origin. Label the rays from the origin covering the points of the Cartesian product anticlockwise by r_1, r_2, \dots, r_m , where $m = |A/A|$.

Our assumptions imply that each ray is incident to $|A|^2/|AA|$ points of $A \times A$. Consider the elements of $A \times A$ as two-dimensional vectors. The sumset $(A \times A) + (A \times A)$ is the same set as $(A + A) \times (A + A)$. We take a subset, S , of this sumset,

$$S = \bigcup_{i=1}^{m-1} (r_i \cap A \times A) + (r_{i+1} \cap A \times A) \subset (A + A) \times (A + A).$$

Simple elementary geometry shows (see the picture below) that the sumsets in the terms are disjoint and each term has $|r_i \cap A \times A| |r_{i+1} \cap A \times A|$ elements. Therefore

$$|S| = |AA|(|A|^2/|AA|)^2 \leq |A + A|^2.$$

After rearranging the inequality we get $|A|^4 \leq |AA||A + A|^2$, as we wanted. Now we will show a rigorous proof based on this observation.

We are going to use the notation of *multiplicative energy*. The name of this quantity comes from a paper of Tao [13], however its discrete version was used earlier, like in [4].

Let A be a finite set of reals. The multiplicative energy of A , denoted by $E(A)$, is given by

$$E(A) = \left| \{(a, b, c, d) \in A^4 \mid \exists \lambda \in \mathbb{R}: (a, b) = (\lambda c, \lambda d)\} \right|.$$

In the notation of Gowers [8], the quantity $E(A)$ counts the number of quadruples in $\log A$.

To establish the proof of Theorem 2.1 we show the following lemma.

Lemma 2.3. *Let A be a finite set of positive real numbers. Then*

$$\frac{E(A)}{[\log |A|]} \leq 4|A + A|^2.$$

Theorem 2.1 follows from Lemma 2.3 via the Cauchy–Schwartz type inequality

$$E(A) \geq \frac{|A|^4}{|AA|}. \quad \square$$

2.2. Proof of Lemma 2.3

Another way of counting $E(A)$ is the following:

$$E(A) = \sum_{x \in A/A} |xA \cap A|^2. \quad (1)$$

The summands on the right hand side can be partitioned into $[\log |A|]$ classes according to the size of $xA \cap A$.

$$E(A) = \sum_{i=0}^{[\log |A|]} \sum_{\substack{x \\ 2^i \leq |xA \cap A| < 2^{i+1}}} |xA \cap A|^2.$$

There is an index, I , that

$$\frac{E(A)}{[\log |A|]} \leq \sum_{\substack{x \\ 2^I \leq |xA \cap A| < 2^{I+1}}} |xA \cap A|^2.$$

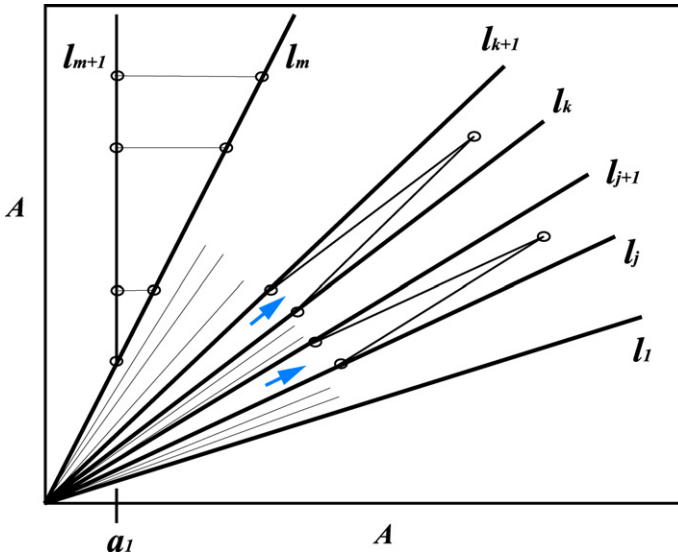


Fig. 1.

Let $D = \{s: 2^I \leq |sA \cap A| < 2^{I+1}\}$, and let $s_1 < s_2 < \dots < s_m$ denote the elements of D , labeled in increasing order,

$$\frac{E(A)}{|\log|A||} \leq \sum_{\substack{x \\ 2^I \leq |xA \cap A| < 2^{I+1}}} |xA \cap A|^2 < m2^{2I+2}. \quad (2)$$

Each line $l_j: y = s_j x$, where $1 \leq j \leq m$, is incident to at least 2^I and less than 2^{I+1} points of $A \times A$. For easier counting we add an extra line to the set, l_{m+1} , the vertical line through the smallest element of A , denoted by a_1 . Line l_{m+1} has $|A|$ points from $A \times A$, however we are considering only the orthogonal projections of the points of l_m . (See Fig. 1.)

The sumset,¹ $(l_i \cap A \times A) + (l_k \cap A \times A)$, $1 \leq j < k \leq m$, has size $|l_i \cap A \times A||l_k \cap A \times A|$, which is between 2^{2I} and 2^{2I+2} . Also, the sumsets along consecutive line pairs are disjoint, i.e.

$$((l_i \cap A \times A) + (l_{i+1} \cap A \times A)) \cap ((l_k \cap A \times A) + (l_{k+1} \cap A \times A)) = \emptyset,$$

for any $1 \leq j < k \leq m$.

The sums are elements of $(A + A) \times (A + A)$, so we have the following inequality,

$$m2^{2I} \leq \left| \bigcup_{i=1}^m (l_i \cap A \times A) + (l_{i+1} \cap A \times A) \right| \leq |A + A|^2.$$

The inequality above with inequality (2) proves the lemma. \square

¹ As customary, by the sum of two points on \mathbb{R}^2 we mean the point which is the sum of their position vectors.

2.3. Remarks

Let A and B be finite sets of reals. The multiplicative energy, $E(A, B)$, is given by

$$E(A, B) = \left| \left\{ (a, b, c, d) \in A \times B \times A \times B \mid \exists \lambda \in \mathbb{R}: (a, b) = (\lambda c, \lambda d) \right\} \right|.$$

In the proof of Lemma 2.3 we did not use the fact that $A = B$, the proof works for the asymmetric case as well. Suppose that $|A| \geq |B|$. With the lower bound on the multiplicative energy

$$E(A, B) \geq \frac{|A|^2 |B|^2}{|AB|}$$

our proof gives the more general inequality

$$\frac{|A|^2 |B|^2}{|AB|} \leq 4 \lceil \log |B| \rceil |A + A| |B + B|.$$

3. Very small productsets

In this section we extend our method from two to higher dimensions. We are going to consider lines though the origin as before, however there is no notion of consecutiveness among these lines in higher dimensions available. We will consider them as points in the projective real space and will find a triangulation of the pointset. The simplices of the triangulation will define the neighbors among the selected lines.

The sum-product bound in Theorem 2.1 is asymmetric. It shows that the productset should be very large if the sumset is small. On the other hand it says almost nothing in the range where the productset is small. For integers, Chang [2] proved that there is a function $\delta(\varepsilon)$ that if $|AA| \leq |A|^{1+\varepsilon}$ then $|A + A| \geq |A|^{2-\delta}$, where $\delta \rightarrow 0$ if $\varepsilon \rightarrow 0$. A similar result is not known for reals. It follows from Elekes' bound [3] (and also from Theorem 2.1) that there is a function $\delta(\varepsilon)$ that if $|AA| \leq |A|^{1+\varepsilon}$ then $|A + A| \geq |A|^{3/2-\delta}$, where $\delta \rightarrow 0$ if $\varepsilon \rightarrow 0$. We prove here a generalization of this bound for k -fold sumsets. For any integer $k \geq 2$ the k -fold subset of A , denoted by kA is the set

$$kA = \{a_1 + a_2 + \cdots + a_k \mid a_1, \dots, a_k \in A\}.$$

Theorem 3.1. *For any integer $k \geq 2$ there is a function $\delta = \delta_k(\varepsilon)$ that if $|AA| \leq |A|^{1+\varepsilon}$ then $|kA| \geq |A|^{2-1/k-\delta}$, where $\delta \rightarrow 0$ if $\varepsilon \rightarrow 0$.*

Proof. We can suppose that A has only positive elements WLOG. Let $|AA| \leq |A|^{1+\varepsilon}$. By a Plünnecke type inequality (Corollary 5.2 [11] or Chapter 6.5 [14]) we have $|A/A| \leq |A|^{1+2\varepsilon}$. Consider the k -fold Cartesian product $A \times A \times \cdots \times A$, denoted by $\times^k A$. It can be covered by no more than $|A/A|^{k-1}$ lines going through the origin. Fig. 2 illustrates the $k = 3$ case. Let H denote the set of lines through the origin containing at least $|A|^{1-2\varepsilon(k-1)}/2$ points of $\times^k A$. With this selection, the lines in H cover at least half of the points in $\times^k A$ since

$$\frac{|A|^{1-2\varepsilon(k-1)}}{2} |A/A|^{k-1} = \frac{|A|^k}{2|A|^{(1+2\varepsilon)(k-1)}} |A/A|^{k-1} \leq \frac{|A|^k}{2}.$$

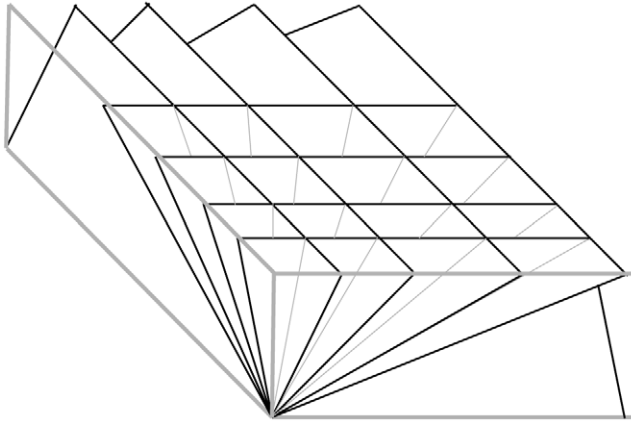


Fig. 2.

As no line has more than $|A|$ points common with $\times^k A$, therefore $|H| \geq |A|^{k-1}/2$. The set of lines, H , represents a set of points, P , in the projective real space $\mathbb{R}P^{k-1}$. Point set P has full dimension $k-1$ as it has a nice symmetry. The symmetry follows from the Cartesian product structure; if a point with coordinates (a_1, \dots, a_k) is in P then the point $(\sigma(a_1), \dots, \sigma(a_k))$ is also in P for any permutation $\sigma \in S_k$. Let us triangulate P . By triangulation we mean a decomposition of the convex hull of P into non-degenerate, $k-1$ -dimensional, simplices such that the intersection of any two is the empty set or a face of both simplices and the vertex set of the triangulation is P . It is not obvious that such triangulation always exists. For the proof we refer to Chapter 7 in [7] or Chapter 2 in [9]. The size of the triangulation (the number of simplices in the triangulation) is at least $|P| - (k-1)$. It is possible that for sets with symmetries like P the maximum triangulation size is much larger, however we were unable to find a better bound. For similar problems about maximum triangulations see [1]. Let $\tau(P)$ be a triangulation of P . We say that k lines $l_1, \dots, l_k \in H$ form a simplex if the corresponding points in P are vertices of a simplex of the triangulation. We use the following notation for this: $\{l_1, \dots, l_k\} \in \tau(P)$. In the two-dimensional case we used that the sumsets of points on consecutive lines are disjoint. Here we are using that the interiors of the simplices are disjoint, therefore sumsets of lines of simplices are also disjoint. Note that we assumed that A is positive, so we are considering convex combinations of vectors with positive coefficients. Let $\{l_1, \dots, l_k\} \in \tau(P)$ and $\{l'_1, \dots, l'_k\} \in \tau(P)$ are two distinct simplices. Then

$$\left(\sum_{i=1}^k l_i \cap \times^k A \right) \cap \left(\sum_{i=1}^k l'_i \cap \times^k A \right) = \emptyset.$$

Also, since the k vectors parallel to the lines $\{l_1, \dots, l_k\} \in \tau(P)$ are linearly independent, all sums are distinct,

$$\left| \sum_{i=1}^k l_i \cap \times^k A \right| = \prod_{i=1}^k |l_i \cap \times^k A|.$$

Now we are ready to put everything together into a sequence of inequalities proving Theorem 3.1,

$$|kA|^k \geq \sum_{\{l_1, \dots, l_k\} \in \tau(P)} \left| \sum_{i=1}^k l_i \cap \times^k A \right| \geq (|A|^{k-1} - k + 1) \prod_{i=1}^k |l_i \cap \times^k A|.$$

Every line is incident to at least $|A|^{1-2\varepsilon(k-1)}/2$ points of $\times^k A$, therefore

$$|kA|^k \geq \frac{|A|^{k-1+k(1-2\varepsilon(k-1))} - (k-1)|A|^{k(1-2\varepsilon(k-1))}}{2^k}.$$

Taking the k th root of both sides we get the result we wanted to show

$$|kA| \geq c_k |A|^{2-1/k-2(k-1)\varepsilon}. \quad \square$$

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